Combinatorial Representations in Structural Analysis

By Offer Shai[1]

Abstract: The work presented in this paper is part of a general research work, during which combinatorial representations based on graph and matroid theories were developed and then applied to different engineering fields. The main combinatorial representations used in this paper are flow and resistance graphs, and resistance matroid representations. The first was applied to the analysis of determinate trusses and the last two were applied to the analysis of indeterminate trusses. This paper gives a description of the representations and the methods embedded within them. The principal methods described in this paper are the conductance cutset method and the resistance circuit method that are mutually dual and are defined for both resistance graph and resistance matroid representations. The present paper shows that the known displacement and force methods are dual since they are the derivatives of the conductance cutset method and resistance circuit method, respectively. The importance of using combinatorial representations in structural mechanics is not only due to the intellectual insight provided by it, but also to its practical applicability. Some practical applications of the approach are reported in this paper, among them even a novel pedagogical framework for structural analysis.

INTRODUCTION

The work reported in this paper is part of a general research during which mathematical models based on discrete mathematics, called combinatorial representations (CR), were developed, the properties in each and the connections between them investigated, and then applied to represent and solve various engineering problems. The representations are based mainly on graph and matroid theories, whereas in the current paper two graph representations and one matroid representation are used. By working in this approach, interesting results have been achieved, a few of which are mentioned below:

- A general perspective on different engineering fields was obtained when the same representation was applied to different problems. For example, the resistance graph representation was applied to analyze both mass-spring-damper systems and indeterminate trusses (Shai and Preiss 1999b).
- New connections between different engineering fields have been achieved by using the connections between the combinatorial representations. For example, a dualism connection between determinate trusses and mechanisms was derived based on the dualism connection between their corresponding representations of flow and potential graphs.
- Known theorems and methods have been derived from the theorems and methods inherent in the representations. For example, based on a theorem inherent in the resistance graph representation, called Tellegen's theorem, Betti's Law and the known method for analyzing the displacement of truss joints have been derived (Shai 2001b).

This paper is a continuation of the approach and it uses CR to give a global perspective on structural analysis and demonstrates it on trusses. It also shows that the known methods, displacement and force, can be derived from the two known methods embedded in the resistance graph representation, conductance cutset method (CCM) and resistance circuit method (RCM), respectively. Furthermore, the present paper shows that this approach enables revealing of the connections between the known methods. This is done by showing on the basis of the dualism between CCM and RCM that the displacement and force methods are dual methods.

The current work deals only with structures, but the underlying approach has been applied also to other fields such as analysis of mechanics and planetary gears (Shai and Preiss 1999a), optimization of structural trusses (Shai 1997), scheduling of robots (Shai 1997), new representations in artificial intelligence (Shai and Preiss 1999b), and others. The use of the graph theory in civil engineering computing has been extensively presented in the literature. For example, some of the subjects covered are analysis of pipe networks (Shimada 1989), computer-assisted mapping for ground surveys (Qi and Lall 1989), a methodology for finding the optimal layout of a detection system in a municipal water network (Kessler et al. 1998), graphic theoretic formulation, which yields a system of ordinary differential equations that describes the dynamic behavior of flows in networks (Onizuka 1986), and a user-optimizing program for traffic assignment based on graph theory concepts (Hatfield 1974).

From this brief review, one can easily comprehend the importance of graph theory to civil engineering computing. This importance is increased when graph theory is augmented by matroid theory, as shown in the present paper.

This paper shows that mathematical graph theory provides a useful generalization, within which all types of axial force structures, either determinate or overdetermined, can be dealt with in a unified way. This generalization is not only intellectually interesting, but is also useful. Various known methods of solving problems in this domain are shown to be particular cases of a more general graph theory problem. This enables useful insights into the methods usually used, thus opening an avenue for further research into fields that have hitherto seemed completely understood. Furthermore, representing the problem as a graph can provide access to mathematically proven methods from graph theory and using them to solve various problems of axial force structures. Also, the approach has been found to be useful as a teaching tool, since learning the graph theory generalization enables students to comprehend a wide range of problems with less effort than by learning each type of problem separately. This paper gives example illustrations for all these aspects.

The following section explains the basics of network graph theory needed for the reader to be able to comprehend the paper. Within the field of graph theory, many different types of graphs have been described and their properties evaluated and published. This paper uses a type of graph called a network graph, and within its context uses two CR, namely the flow and resistance graph representations.

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This paper shows how to represent a determinate truss as a flow graph, and then to analyze it. Also, the paper shows how an indeterminate truss can be represented as a resistance graph. The suitable algorithm resulting from the properties of the graph representation is then applied to indeterminate trusses. The first use of graph theory for the analysis of elastic networks was done by Kron (1963), who used the analogy between electrical networks and elastic structures. To ease the computational burden, Kron developed the so-called ''diakoptics.'' Since then, other works have been published, such as Lind (1962), who showed how to represent trusses by graphs and then analyze them. Fenves and Branin (1963), who developed a method based on graphs and networks for the formulation of structural analysis and on its base, developed the software program STRESS (Fenves et al. 1965). Graph theory has also been used in finite-element analysis (Shanghui and Guohua 1984). Structural analysis and optimization using graph and matroid theories have been performed by Kaveh (1991, 1997). A comprehensive work on structural rigidity using matroid theory appears in Recski (1989).

An approach that deals with structural mechanics in general is less known in the literature. However, Bjorke has published works (Wang and Bjorke 1989, 1991) where he established a unified theory to represent a manufacturing system. He found that network theory is probably the best foundation for this purpose. The proper equation for the specific problem was obtained by using a technique that is based on Roth's (1955) diagram.

The approach reported in this paper is different. After introduction to network graphs and definitions of the graph representations, two general and mutually dual graph methods called resistance circuit method (RCM) and conductance cutset method (CCM) are developed only on the basis of resistance graph properties. These methods are known in the literature for 1D (electrical) systems, whereas in this paper it is shown how to expand them for multidimensional engineering problems. These two general methods are then applied to the analysis of trusses.

NETWORK GRAPHS

The combinatorial representations that are used in this paper are graphs; therefore, the current section provides the reader with a brief survey on the graph theory terminology. More details can be found in Shai and Preiss (1999a) or in books on graph theory, such as Swamy and Thulasiraman (1981).

A graph is defined by the ordered pair $G = \langle \mathbf{V}, \mathbf{E} \rangle$, where $\mathbf{V}$ is the vertex set and $\mathbf{E}$ is the edge set, and every edge is defined by its two end vertices. If each edge in the graph is assigned a direction, then the graph is known as a directed graph. The directed graph is a network graph if each edge and vertex has properties of flow and potential, respectively.

For convenience, the paper uses linetype attributes, which are

- A solid line represents an edge with unknown value of flow or potential difference.
- A bold line represents an edge for which the flow or potential difference is known.
- A dashed line represents a chord, which is an edge not included in the spanning tree. If the value of flow in the chord is known, then it is both dashed and bold.
- A double line represents a branch of a spanning tree.

Given a connected network graph, choosing a spanning tree within it defines its branches and chords. In order to deal with the graph representations used in this paper, one should first define cutset and circuit matrices in their vector and scalar forms.

A cutset in a connected graph is a minimal set of edges whose removal results in a disconnected graph. It can be proved that a cutset separates the graph into two components (maximal connected subgraphs). When a cutset includes exactly one branch of the spanning tree it is called a ''fundamental cutset.'' This paper deals mostly with fundamental cutsets, hence for brevity they will be called cutsets. Each cutset is defined by the corresponding branch and is labeled with its branch index. The direction of the cutset is defined by the direction of its branch, as shown in Fig. 1(a).

The vector cutset matrix $\bar{\mathbf{Q}}$ is a matrix that describes all the cutsets but contains only topological information. The matrix has $e(G)$ columns (corresponding to the edges of the graph) and $\mathbf{v}(G) - 1$ rows (corresponding to the cutsets or branches that define them). The value of the matrix element $[\bar{\mathbf{Q}}]_{ij}$ may be $+1$, 0, or -1. It is $+1$ if edge j is included in the cutset that is defined by branch i and has the same orientation as the cutset, -1 if it has the opposite orientation, and 0 if it is not included in the cutset. The vector cutset matrix of the graph in Fig. 1(a) is shown in Fig. 1(b).

The scalar cutset matrix $\mathbf{Q}$ is obtained from the vector cutset matrix $\bar{\mathbf{Q}}$ by multiplying each column with a unit vector in the direction of the edge to which it corresponds. For example, the scalar cutset matrix of the graph in Fig. 2(a) is given in Fig. 2(b).

A circuit is a set of edges that form a closed path. A circuit is called a fundamental circuit if it includes exactly one chord. This paper deals only with fundamental circuits, and for brevity they will be called circuits. Each circuit will be labeled with the index of the chord that defines it. The direction of the circuit is defined by the direction of its chord, as shown in Fig. 3(a).

The vector circuit matrix $\bar{\mathbf{B}}$, demonstrated in Fig. 3, has $e(G)$

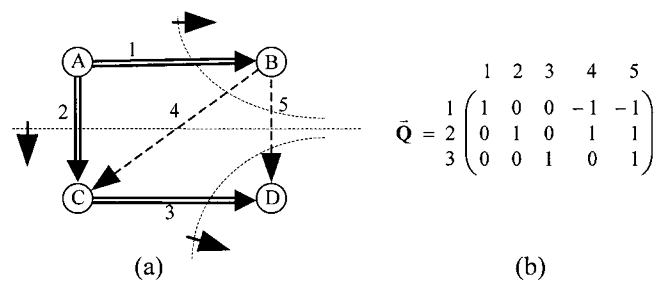


FIG. 1. Example of Vector Cutset Matrix: (a) Cutsets of Graph; (b) Vector Cutset Matrix

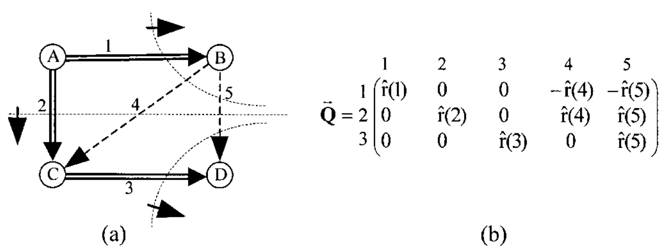


FIG. 2. Example of Scalar Cutset Matrix: (a) Cutsets of Graph; (b) Scalar Cutset Matrix

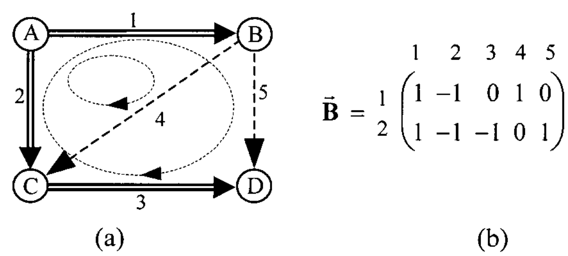


FIG. 3. Example of Vector Circuit Matrix: (a) Circuits of Graph; (b) Corresponding Vector Circuit Matrix

columns as for the vector cutset matrix and $e(G) - \mathbf{v}(G) + 1$ rows corresponding to the circuits. Each circuit is defined by a chord, therefore the number of rows is equal to the number of chords defined by the spanning tree. The element $[\bar{\mathbf{B}}]_{ij} = +1$ if edge j is included in the circuit defined by chord i, and has the same orientation as the circuit, -1 if it has the opposite orientation, and 0 otherwise.

Every edge is assigned a vector called the flow, designated by $\bar{F}(e)$. Flow can correspond to a force, flow of liquid, money, goods, information, or the like. In control theory, this is called the "through variable," but the word "flow" is more suitable for the work reported here. In this paper the flow in edge e is interpreted as the force in the corresponding rod in the structure.

Every vertex is assigned a vector called the potential and designated by $\bar{\pi}(\mathbf{v})$. The potential may represent a physical quantity such as displacement, pressure, or voltage, but it can also be used for other attributes. For instance, in the shortest path algorithm it represents the lower bound of the distance (or the sum of the edge weights) from the current vertex to the target vertex (Shai 1997). In this paper it represents the displacement of a structure joint that corresponds to vertex $\mathbf{v}$. The potentials of the vertices of edge $e = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle$ define the potential difference of the edge, as follows:

$$\bar{\Delta}(e) = \bar{\pi}(\mathbf{v}_2) - \bar{\pi}(\mathbf{v}_1) \quad (1)$$

The potential difference is known in control theory as the "across variable."

FLOW GRAPH REPRESENTATION

A network graph G is a flow graph, designated by G_F, if the flows in the edges are independent of the potential differences and satisfy the flow law, stated as follows: "The vector sum of the flows in every cutset of G is equal to zero." The flow law may be regarded as a generalization of the well-known Kirchhoff's current law. Note that Kirchhoff's current law is restricted only to one dimension which is appropriate for electrical circuits, while the flow law can be multidimensional, thus it can be used for structures and other engineering systems whose dimension is two or three. The matrix form of the flow law is

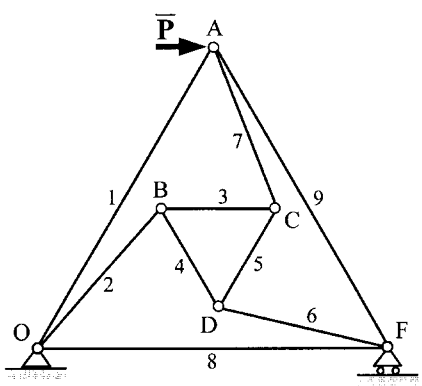
$$\bar{\mathbf{Q}} \cdot \bar{\mathbf{F}} = \mathbf{0} \quad (2)$$

where $\bar{\mathbf{F}}$ = vector of the flows, or the flow vector.

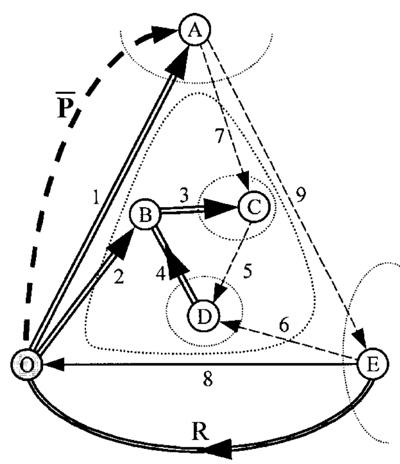
The flow graph representation can be used to represent various engineering systems, such as simple electrical circuits, mass-cable systems in force equilibrium, and so forth. This paper uses the flow graph to represent statically determinate trusses.

The steps for representing the truss by a flow graph are:

1. Create a vertex in the graph for every pinned joint of the truss.
2. For every rod create an edge in the graph, called a "truss edge"; its end vertices correspond to the joints that connect the corresponding rod to the truss. Assign an arbitrary orientation to each truss edge and a unit vector $\hat{\mathbf{r}}(e)$ directed from the tail joint to the head joint. The engineering meaning of the flow in the edge corresponding to a rod is the force applied on the head vertex (joint) by the rod in the direction of the unit vector $\hat{\mathbf{r}}(e)$, which is, of course, equal to the force that the tail vertex (joint) applies on the rod. If the flow in the edge is positive, then the rod is in the state of compression, otherwise it is in a state of tension.
3. Add an extra vertex called the "reference vertex" to the graph. The reference vertex is a generalization of the "ground" in electrical circuits or of "datum" in structural sketches.
4. For each externally applied force and reaction, add an edge as follows. For each externally applied force a



(a)



(b)

$$\begin{array}{c} Rx \\ Ry \\ 1x \\ 1y \\ 2x \\ 2y \\ 3x \\ 3y \\ 4x \\ 4y \end{array}
\begin{pmatrix}
\cos(270^\circ) & 0 & 0 & 0 & 0 & 0 & \cos(170^\circ) & 0 & \cos(0^\circ) & -\cos(300^\circ) \\
\sin(270^\circ) & 0 & 0 & 0 & 0 & 0 & \sin(170^\circ) & 0 & \sin(0^\circ) & -\sin(300^\circ) \\
0 & \cos(60^\circ) & 0 & 0 & 0 & 0 & 0 & -\cos(230^\circ) & 0 & -\cos(300^\circ) \\
0 & \sin(60^\circ) & 0 & 0 & 0 & 0 & 0 & -\sin(230^\circ) & 0 & -\sin(300^\circ) \\
0 & 0 & \cos(50^\circ) & 0 & 0 & 0 & \cos(170^\circ) & \cos(230^\circ) & 0 & 0 \\
0 & 0 & \sin(50^\circ) & 0 & 0 & 0 & \sin(170^\circ) & \sin(230^\circ) & 0 & 0 \\
0 & 0 & 0 & \cos(0^\circ) & 0 & -\cos(240^\circ) & 0 & \cos(230^\circ) & 0 & 0 \\
0 & 0 & 0 & \sin(0^\circ) & 0 & -\sin(240^\circ) & 0 & \sin(230^\circ) & 0 & 0 \\
0 & 0 & 0 & 0 & \cos(120^\circ) & -\cos(240^\circ) & -\cos(170^\circ) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sin(120^\circ) & -\sin(240^\circ) & -\sin(170^\circ) & 0 & 0 & 0
\end{pmatrix}
\cdot
\begin{pmatrix} F_R \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \end{pmatrix}
= -
\begin{pmatrix} 0 \\ 0 \\ \cos(0^\circ) \\ \sin(0^\circ) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} P$$

(c)

FIG. 4. Example for Analysis of Determinate Truss Using Flow Graph Representation: (a) Statically Determinate Plane Truss; (b) Corresponding Flow Graph; (c) Force Analysis Equations in Scalar Cutset Matrix Form

"flow source edge" is added. Its tail vertex is the reference vertex and the head vertex is the vertex corresponding to the joint upon which the external force acts. Because of the requirements of the analysis algorithm that are given later, the latter edges should always be chosen to be chords. Since flow source edges are chords and the flows in them are known, they appear in the graph as bold and dashed lines. For each roller support reaction, a "reaction edge" is added. Its tail vertex is the vertex corresponding to the joint upon which the reaction acts and the head vertex is the reference vertex. The reaction edge is assigned an angle equal to the angle of the reaction. On the other hand, for each hinged support, two reaction edges are added, the first having the corresponding angle equal to 180° and the second to 270°.

In trusses, the sum of forces applied on each joint is equal to zero. In the terminology of the flow graph representation, this means that the flows in the graph that corresponds to the truss, satisfy the flow law. Thus, the force analysis process of the truss is transformed into a search for flows that satisfy the flow law in the corresponding graph. The flow of forces can be thought of as originating at the reference vertex, then flowing through the source edges representing the external forces, then flowing through the truss rods, and then returning via the reaction edges back to the reference vertex. An example of a truss, its corresponding graph, and the analysis equations written according to (2) are given in Fig. 4.

POTENTIAL GRAPH REPRESENTATION

This combinatorial representation is briefly mentioned in this paper, therefore it will be explained briefly. A network graph G is called a potential graph, designated by G_Δ, if the potential differences in its edges are independent of the flows and satisfy the potential law, which states "The vector sum of potential differences in every circuit of the graph is equal to zero." The matrix form of the potential law is

$$\bar{\mathbf{B}} \cdot \bar{\boldsymbol{\Delta}} = \mathbf{0} \tag{3}$$

where $\bar{\boldsymbol{\Delta}}$ = vector of potential differences or the potential difference vector. The use of potential graph for velocity analysis of mechanisms is described, where it is proved to be dual to the flow graph representation. Following this relation it was proved that mechanisms and determinate trusses are also dual, as illustrated in a later section.

RESISTANCE GRAPH REPRESENTATION

Description

The resistance graph is a generalization of the flow and the potential graphs. The resistance graph is a network graph, designated by G_R, where there are edges with dependence between the flow and the potential difference. Such dependence is characterized by either a scalar or a matrix. The scalar is used if there is an explicit dependence between the vector magnitudes of the flow and potential difference, otherwise the matrix is used. For both scalar and matrix possibilities there are two presentations: (1) resistance [designated by $R(e)$ and $\mathbf{R}(e)$, respectively]; and (2) conductance (designated by K and $\mathbf{K}$, respectively), as follows:

$$|\bar{\Delta}(e)| = R(e) \cdot |\bar{F}(e)|; \quad |\bar{F}(e)| = K(e) \cdot |\bar{\Delta}(e)| \tag{4}$$

$$\bar{\Delta}(e) = \mathbf{R}(e) \cdot \bar{F}(e); \quad \bar{F}(e) = \mathbf{K}(e) \cdot \bar{\Delta}(e) \tag{5}$$

where $\bar{\Delta}(e)$ = potential difference in edge e; and $\bar{F}(e)$ = flow. Flows and potential differences of the resistance graph must satisfy the flow and potential laws, respectively.

When dealing with resistance graph representation, an important theorem from graph theory, called the orthogonality principle, becomes essential. The orthogonality principle states that vector cutset and circuit matrices are orthogonal

$$\bar{\mathbf{B}} \cdot \bar{\mathbf{Q}}^t = \mathbf{0} \tag{6}$$

From this principle the following equations can be established (Swamy and Thulasiraman 1981):

$$\bar{\boldsymbol{\Delta}} = \bar{\mathbf{Q}}^t \cdot \bar{\boldsymbol{\Delta}}_T \tag{7}$$

$$\bar{\mathbf{F}} = \bar{\mathbf{B}}^t \cdot \bar{\mathbf{F}}_T \tag{8}$$

The edges in the resistance graph are divided into three principal groups: flow sources, potential difference sources, and resistance edges. Flow sources, denoted by bold dashed lines, are edges for which the value of the flow is known and is independent of the potential difference. Potential difference sources, denoted by bold solid lines, are the edges in which the potential difference is known and is independent of the flow in that edge. Resistance edges, denoted by black solid lines, are the edges at which there is a dependence between the flow and the potential difference.

Conductance Cutset Method for Solving Resistance Graphs

The analysis problem for the resistance graph is that given the flows in the flow sources, the potential differences in the potential difference sources, and the resistances (or conductances) of the resistance edges, find the flows and potential differences in all the edges of the graph. The obvious method for solving the resistance graphs is to write all the equations based on (2)–(5), and then to solve them simultaneously. This method has a high computational complexity, therefore the current and the next sections show more efficient analysis methods based on graph theory theorems. The CCM will be explained first.

In the graph there can be no circuits consisting of only potential difference sources. If such circuits of sources existed, then, by the potential law, there would be linear dependence between the potential differences in these sources. Such a dependence violates the definition of potential difference sources.

For the same reason in the resistance graph there can be no cutset of the flow sources consisting of only the flow sources. Therefore, it can be proven (Swamy and Thulasiraman 1981) that there exists in the resistance graph, a spanning tree containing all the potential difference sources and no flow sources. Finding such a spanning tree is the starting point for the development of both CCM and RCM.

Eq. (2) is now rewritten, dividing both the vector cutset matrix and flow vector in accordance to the types of edges

$$\begin{array}{c} \\ \Delta \\ T' \end{array} \begin{array}{c} \;\;\Delta \quad\;\; R \qquad\; P \\ \begin{pmatrix} \mathbf{I} & \bar{\mathbf{Q}}_{\Delta R} & \bar{\mathbf{Q}}_{\Delta P} \\ \mathbf{0} & \bar{\mathbf{Q}}_{T'R} & \bar{\mathbf{Q}}_{T'P} \end{pmatrix} \end{array} \cdot \begin{pmatrix} \bar{\mathbf{F}}_\Delta \\ \bar{\mathbf{F}}_R \\ \bar{\mathbf{F}}_P \end{pmatrix} = \mathbf{0} \Rightarrow \bar{\mathbf{Q}}_{T'R}\bar{\mathbf{F}}_R = -\bar{\mathbf{Q}}_{T'P}\bar{\mathbf{F}}_P \tag{9}$$

where Δ and P = edges corresponding to the potential difference and flow sources, respectively; R = remaining edges of the graph that are the edges with resistance; and T' = spanning tree branches that are not sources.

The crucial property of the potential differences in the resistance graph is that the potential differences in the branches of the spanning tree uniquely determine the potential differences in the chords. This can be formalized as follows (Swamy and Thulasiraman 1981):

$$\bar{\mathbf{\Delta}}_C = -\bar{\mathbf{B}}_T\bar{\mathbf{\Delta}}_T \Rightarrow \bar{\mathbf{\Delta}} = \mathbf{Q}^t\bar{\mathbf{\Delta}}_T \Rightarrow \begin{pmatrix} \bar{\mathbf{\Delta}}_\Delta \\ \bar{\mathbf{\Delta}}_R \\ \bar{\mathbf{\Delta}}_P \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \bar{\mathbf{Q}}^t_{\Delta R} & \bar{\mathbf{Q}}^t_{T'R} \\ \bar{\mathbf{Q}}^t_{\Delta P} & \bar{\mathbf{Q}}^t_{T'P} \end{pmatrix} \cdot \begin{pmatrix} \bar{\mathbf{\Delta}}_\Delta \\ \bar{\mathbf{\Delta}}_{T'} \end{pmatrix}$$

$$\Rightarrow \bar{\mathbf{\Delta}}_R = \bar{\mathbf{Q}}^t_{\Delta R}\bar{\mathbf{\Delta}}_\Delta + \bar{\mathbf{Q}}^t_{T'R}\bar{\mathbf{\Delta}}_{T'} \quad (10)$$

As shown in Swamy and Thulasiraman (1981), combining (5), (9), and (10) results in

$$(\bar{\mathbf{Q}}_{T'R}\cdot\mathbf{K}_R\cdot\bar{\mathbf{Q}}^t_{T'R})\cdot\bar{\mathbf{\Delta}}_{T'} = -(\bar{\mathbf{Q}}_{T'R}\cdot\mathbf{K}_R\cdot\bar{\mathbf{Q}}^t_{\Delta R})\cdot\bar{\mathbf{\Delta}}_\Delta - \bar{\mathbf{Q}}_{T'P}\cdot\bar{\mathbf{F}}_P \quad (11)$$

where $\mathbf{K}_R$ = square diagonal matrix whose components correspond to the conductances in the resistance edges.

The values of the elements of the matrix products can be derived on the basis of linear algebra considerations, as follows. $[\bar{\mathbf{Q}}_{T'R}\cdot\mathbf{K}_R\cdot\bar{\mathbf{Q}}^t_{T'R}]_{ij}$ is the sum of conductances of the edges that belong to both cutsets i and j defined by branches with resistance; the sign of the conductance is taken positive if it is similarly directed relative to both cutsets, negative otherwise. $[\bar{\mathbf{Q}}_{T'R}\cdot\mathbf{K}_R\cdot\bar{\mathbf{Q}}^t_{\Delta R}]_{ij}$ is also the sum of conductances of the edges that belong to both cutsets i and j, but this time j is a branch that is a potential difference source, while i is a branch with resistance, as before.

After solving (11), all the potential differences in the branches are known. All the potential differences in the graph are obtained by applying (7), after which all the flows in the graph are obtained by (4) or (5).

RCM for Solving Resistance Graphs

The CCM for analysis of resistance graphs, which was developed in the previous section, exploits the rule that the potential differences in the branches of the spanning tree uniquely determine the potential differences in all the graph edges. Therefore using CCM, a set of linear equations is obtained, where the potential differences in the branches are the only unknowns.

The RCM shown in this section is similar to CCM and is actually its dual method, being derivable from it by using only the dualism connection explained later in the section. RCM allows one to obtain the set of linear equations whose only unknowns correspond to the flows in the chords of the graph. It takes advantage of a basic rule that states that the flows in the chords of the graph uniquely determine the flows in all the graph edges (Swamy and Thulasiraman 1981).

The first step in the RCM is choosing a spanning tree. As explained in the previous section, the spanning tree should contain all the potential difference sources and should not contain any flow sources. Further development steps are exactly dual to those made during the development of CCM and can be found in Shai (1999).

The final formula of RCM is then

$$(\bar{\mathbf{B}}_{C'R}\cdot\mathbf{R}_R\cdot\bar{\mathbf{B}}^t_{C'R})\cdot\bar{\mathbf{F}}_{C'} = -(\bar{\mathbf{B}}_{C'R}\cdot\mathbf{R}_R\cdot\bar{\mathbf{B}}^t_{PR})\cdot\bar{\mathbf{F}}_P - \bar{\mathbf{B}}_{C'P}\cdot\bar{\mathbf{\Delta}}_\Delta \quad (12)$$

where $\mathbf{R}_R$ = square diagonal matrix, the components of which correspond to the resistances in the resistance edges. The elements of the matrix products can be derived on the basis of linear algebra considerations as follows. $[\bar{\mathbf{B}}_{C'R}\cdot\mathbf{R}_R\cdot\bar{\mathbf{B}}_{C'R}]_{ij}$ is the sum of the resistances of the edges that belong to both the circuits i and j, defined by the chords with resistance. The sign of the resistance is positive if the corresponding edge is directed similarly in relation to both circuits, negative otherwise. $[\bar{\mathbf{B}}_{C'R}\cdot\mathbf{R}_R\cdot\bar{\mathbf{B}}^t_{PR}]_{ij}$ is calculated the same way, except that circuit j is now defined by the chord that is the flow source, while i is the chord with resistance, as before.

Eq. (12) is actually the set of linear equations, the unknowns of which are the flows in the resistance chords of the graph. After solving it, all the flows in the graph are obtained by using (8) and after that all the potential differences in the graph are obtained by (4) or (5).

As was already pointed out above in this section, CCM and RCM are mutually dual methods. This means that the CCM applied to resistance graph G is equivalent to the RCM applied to the resistance graph G*, which is the dual graph of G.

Resistance Graph Representation for Multidimensional Trusses

The interpretation of the flows in the edges of the resistance graph representing a truss remains similar to that of the flow graph representing a determinate truss. The potential of the vertex of the resistance graph representing a truss is equal to the displacement vector of the corresponding pinned joint of the truss. Consequently, the potential differences in the graph edges are equal to the relative displacements of the end joints of the corresponding rods.

One now turns to obtaining the conductances of the graph edges. Let dim($\bar{\mathbf{F}}$) be the dimension of the forces in the truss. The explanation provided here is for plane trusses [dim($\bar{\mathbf{F}}$) = 2], but the approach is valid for three dimensions as well, and can easily be extended.

Eq. (5) can be rewritten as

$$\bar{\mathbf{F}} = \mathbf{K}_R\cdot\bar{\mathbf{\Delta}} \quad (13)$$

where $\mathbf{K}_R$ is built from the conductivity matrices of the resistance edges, each being a square matrix of size dim($\bar{\mathbf{F}}$) × dim($\bar{\mathbf{F}}$) and is derived as shown in Fig. 5.

Let $\bar{\mathbf{\Delta}}_i(e)$ correspond to the potential difference between the two end vertices of edge e in the coordinate axis i. Under the small deflection assumption (West 1993), one can obtain from Fig. 5—which shows the initial and the deformed states of the rod—the following equation describing the scalar magnitude of the potential difference as a function of its coordinate components:

$$|\bar{\Delta}(e)| = \mathbf{\Delta}_x(e)\cdot\cos\alpha + \mathbf{\Delta}_y(e)\cdot\sin\alpha \quad (14)$$

where α = angle of the element.

Combining (5) and (14) gives

$$\bar{\mathbf{F}}(e) = \begin{pmatrix} F_x(e) \\ F_y(e) \end{pmatrix} = K(e)\cdot\begin{pmatrix} \cos^2\alpha & \sin\alpha\cdot\cos\alpha \\ \sin\alpha\cdot\cos\alpha & \sin^2\alpha \end{pmatrix}\cdot\begin{pmatrix} \mathbf{\Delta}_x(e) \\ \mathbf{\Delta}_y(e) \end{pmatrix}$$

$$= \mathbf{K}(e)\begin{pmatrix} \mathbf{\Delta}_x(e) \\ \mathbf{\Delta}_y(e) \end{pmatrix} \quad (15)$$

Two 2D conductance matrix of the graph edges [designated by $\mathbf{G}(e)$] is the product of the constant conductivity and the transformation matrix. Thus, for the edges corresponding to truss rods, the constant conductivity is equal to the rod stiffness. For edges corresponding to hinged support reactions, the constant should be taken as 0, since there is no dependence between the displacement of the support and the reaction force.

In indeterminate trusses the forces in the rods cannot be determined by the laws of statics alone, so one must also consider the compatibility conditions. In the terminology of the graph representation, this means that the resistance graph representing the indeterminate truss should be analyzed by using the flow and potential laws simultaneously. Thus, the process

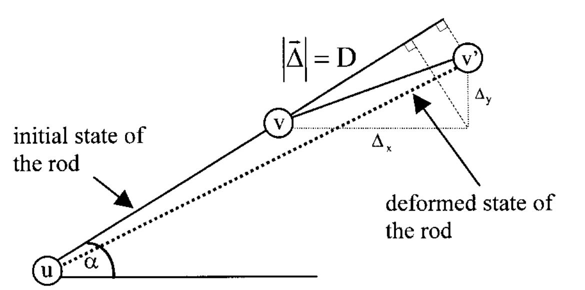


FIG. 5. Rod Deformation

TABLE 1. Types of Edges in Resistance Graph Representation of Indeterminate Truss

Type of edge	Conductance of edge
Truss rod—Resistance edge with finite conductance	$\frac{A(e)\cdot E(e)}{L(e)}\cdot\begin{pmatrix}\cos^2\alpha & \sin\alpha\cdot\cos\alpha \\ \sin\alpha\cdot\cos\alpha & \sin^2\alpha\end{pmatrix}$ where $A(e)$, $E(e)$, and $L(e)$ are cross-sectional area, modulus of elasticity, and length of rod e, respectively
Fixed supports and roller supports on inclined surface	Zero
External force applied on the truss—Flow source edge	No constant conductance

of building the resistance graph corresponding to an indeterminate truss can be summarized as follows.

Build a graph following the same steps as explained in a previous section for building the flow graph of a determinate truss. The flow graph becomes a resistance graph when one assigns resistances (or conductances) to all its edges, as shown in Table 1, that produce potential differences subject to the potential law.

Analysis Process Using CCM

The process is based on applying the CCM to the resistance graph representation of the indeterminate truss. The first step is choosing a suitable spanning tree. The reaction edges have properties similar to those of the potential difference sources since they preserve a zero potential difference along one or two axes. Hence, as was explained above, the spanning tree must include all the reaction edges, but it should not include the flow source edges.

Once the equations of CCM have been obtained, the compatibility conditions on the reaction edges are to be satisfied. For each reaction edge corresponding to a roller support, there is a constant relation between the x- and y-components of the potential difference that is equal to the tangent of the plane inclination angle. This relation is taken into account by replacing one of the components by the other multiplied with the relation coefficient. This is equivalent to replacing two corresponding columns in the conductance cutset matrix by one equal to their linear combination. The resultant matrix would possess one linearly dependent row that should be removed. An example of an indeterminate truss, its corresponding graph and the spanning tree, is shown in Fig. 6.

The CCM gives the following set of equations:

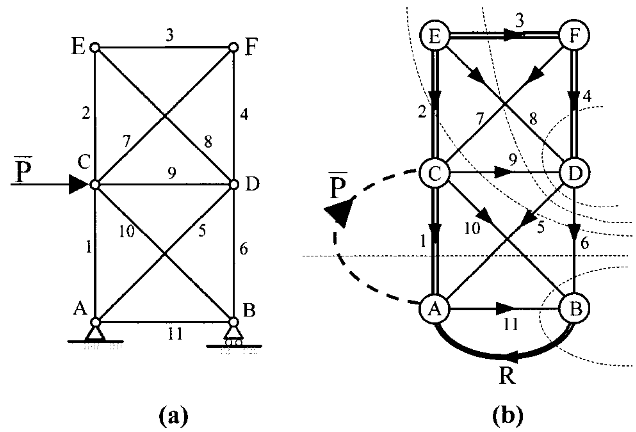


FIG. 6. (a) Statically Indeterminate Truss; (b) Resistance Graph

Now, suppose, for example, that the roller support R in the truss of Fig. 6 was located on a surface inclined by an angle α, i.e., there is an inclined roller support instead of a horizontal one. In this case, an adjustment should be done to the analysis equations. The column corresponding to $\boldsymbol{\Delta}_{Rx}$ in the conductance cutset matrix is multiplied by tan α and is added to the column corresponding to Δ_{Ry}. This way the two unknown Δ_{Ry} and Δ_{Rx} are replaced only by Δ_{Ry}. Now removing one dependent equation results in a new equation set that can be analyzed.

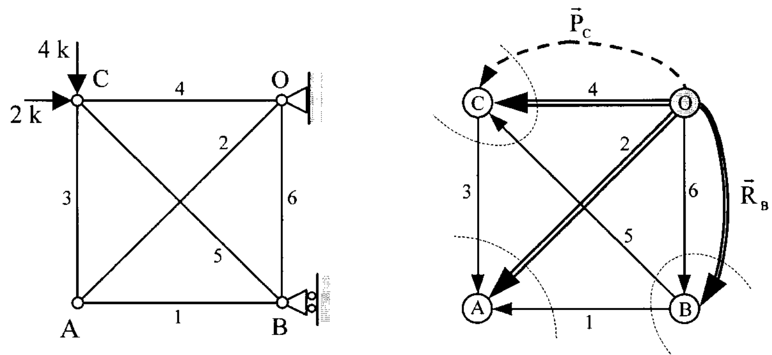
Relation between Conductance Cutset Method for Analyzing Indeterminate Trusses and Other Known Methods

Many methods for analyzing indeterminate trusses have been reported in the literature; most of them are based on virtual work and the minimum energy principle. In the example presented in Fig. 7, all the cutsets are chosen to contain exactly one vertex in one of the two sides of the cutset, thus the corresponding cutset matrix is equal to the incidence matrix, a well-known matrix in graph theory literature (Fenves 1966; Deo 1974).

Choosing the incidence matrix instead of the cutset matrix makes it possible to reveal the relation between the CCM and

$$
\begin{array}{c} \\ R \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{cccccc} R & 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{pmatrix}
K_R+K_6+K_8+K_9+K_{13} & K_9+K_{13} & -K_6-K_8-K_9 & -K_8-K_9 & -K_8-K_9 & 0 \\
K_9+K_{13} & K_1+K_9+K_{13} & -K_9 & -K_9 & -K_9 & 0 \\
-K_6-K_8-K_9 & -K_9 & K_2+K_6+K_8+K_9+K_{10}+K_{11}+K_{12} & K_8+K_9+K_{10}+K_{11}+K_{12} & K_8+K_9+K_{11}+K_{12} & -K_{10}-K_{11} \\
-K_8-K_9 & -K_9 & K_8+K_9+K_{10}+K_{11}+K_{12} & K_3+K_7+K_8+K_9+K_{10}+K_{11}+K_{12} & K_7+K_8+K_9+K_{11}+K_{12} & -K_{10}-K_{11} \\
-K_8-K_9 & -K_9 & K_8+K_9+K_{11}+K_{12} & K_7+K_8+K_9+K_{11}+K_{12} & K_4+K_7+K_8+K_9+K_{11}+K_{12} & -K_{11} \\
0 & 0 & -K_{10}-K_{11} & -K_{10}-K_{11} & -K_{11} & K_5+K_{10}+K_{11}
\end{pmatrix}
\end{array}
$$

$$
\cdot\begin{pmatrix} \Delta_{Rx} \\ \Delta_{Ry} \\ \Delta_{1x} \\ \Delta_{1y} \\ \Delta_{2x} \\ \Delta_{2y} \\ \Delta_{3x} \\ \Delta_{3y} \\ \Delta_{4x} \\ \Delta_{4y} \\ \Delta_{5x} \\ \Delta_{5y} \end{pmatrix} = -\begin{pmatrix} -\cos(0) \\ -\sin(0) \\ 0 \\ 0 \\ \cos(0) \\ \sin(0) \\ \cos(0) \\ \sin(0) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\cdot P
$$



(a) (b)

$$\mathbf{K}_{R_B} = 0\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \quad \mathbf{K}_1 = \frac{AE}{10}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \quad \mathbf{K}_2 = \frac{AE}{\sqrt{2}\cdot 10}\begin{pmatrix}0.5 & 0.5\\0.5 & 0.5\end{pmatrix} \quad \mathbf{K}_3 = \frac{AE}{10}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

$$\mathbf{K}_4 = \frac{AE}{10}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \quad \mathbf{K}_5 = \frac{AE}{\sqrt{2}\cdot 10}\begin{pmatrix}0.5 & -0.5\\-0.5 & 0.5\end{pmatrix} \quad \mathbf{K}_6 = \frac{AE}{10}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

$$\begin{pmatrix}\mathbf{K}_{R_B}+\mathbf{K}_1+\mathbf{K}_5+\mathbf{K}_6 & -\mathbf{K}_1 & -\mathbf{K}_5\\ -\mathbf{K}_1 & \mathbf{K}_2+\mathbf{K}_1+\mathbf{K}_3 & -\mathbf{K}_3\\ -\mathbf{K}_5 & -\mathbf{K}_3 & \mathbf{K}_4+\mathbf{K}_3+\mathbf{K}_5\end{pmatrix}\cdot\begin{pmatrix}\overline{\Delta}_{R_B}\\ \overline{\Delta}_2\\ \overline{\Delta}_4\end{pmatrix} = -\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\overline{P}$$

(c)

FIG. 7. Example of Relation between CCM and Displacement Method: (a) Indeterminate Truss; (b) Corresponding Graph; (c) Corresponding Equations

the displacement method. In this section it is shown that the stiffness matrix of the displacement method (Hibbeler 1984) is the same as the conductance cutset matrix obtained from the resistance graph, as shown in Fig. 7. It is interesting to notice that the number of elements of value zero in the matrix obtained using the displacement method is fixed for any numbering of the vertices, while using CCM this number varies according to the chosen spanning tree.

DERIVING DISPLACEMENT METHOD FROM RESISTANCE GRAPH REPRESENTATION

To derive the displacement method, one starts with the incidence matrix (Deo 1974), the rows of which are linearly dependent on the rows of the vector cutset matrix. The flow law can be written by using the incidence matrix as follows:

$$\mathbf{A}\overline{\mathbf{F}}_R = -\mathbf{A}_P\overline{\mathbf{P}} \tag{16}$$

Since any resistance edge $e = \langle \mathbf{u}, \mathbf{v}\rangle$ satisfies $\overline{\mathbf{F}}(e) = \mathbf{G}(e)\overline{\mathbf{\Delta}}(e)$, where $\mathbf{G}(e)$ is defined in (5), (16) becomes

$$\mathbf{A}\mathbf{K}_R\overline{\mathbf{\Delta}}_R = -\mathbf{A}_P\overline{\mathbf{P}} \tag{17}$$

The potential difference in an edge is equal to the difference between the potentials of the end vertices [(1)], or in the matrix form

$$\overline{\mathbf{\Delta}}_R = \mathbf{A}^t\overline{\boldsymbol{\pi}} \tag{18}$$

that gives

$$\mathbf{A}\mathbf{K}_R\mathbf{A}^t\overline{\boldsymbol{\pi}} = -\mathbf{A}_P\overline{\mathbf{P}} \tag{19}$$

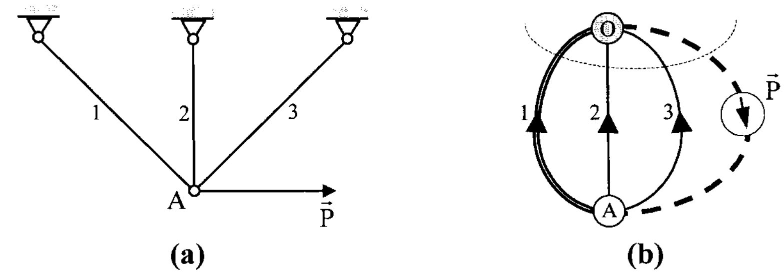
The matrix $\mathbf{A}\mathbf{K}_R\mathbf{A}^t$ is actually the "stiffness matrix," and the minus sign of the element $[\mathbf{A}\mathbf{K}_R\mathbf{A}^t]_{ij}$ is equal to the conductance of the rod that meets both joints i and j (in the case $i = j$ it equals the sum of conductances of all the rods meeting joint i). It is important to notice that a dual method of the CCM for analyzing indeterminate trusses does not exist in the graph representation, since the determinant of the conductance matrix $\mathbf{K}(e)$ of each rod is equal to zero and therefore has no inverse. However, it will be shown later that a dual method to the CCM for trusses does exist when the trusses are represented by the resistance matroid representation (RMR).

RMR FOR MULTIDIMENSIONAL INDETERMINATE TRUSS

Graph theory was used above to derive the displacement method from CCM in the resistance graph representation. However, this approach has been shown to have its limitations, one of which is the impossibility of applying the RCM to trusses, i.e., the method dual to CCM. It is known from the literature that matroid theory, whose definitions and properties are given in the Appendix, is a generalization of graph theory. Therefore, representing engineering systems by matroid theory enables one to obtain a more general perspective. Such a generalization is demonstrated in this section by representing indeterminate trusses with RMR. Doing so enables one to reveal the duality relation between the force and displacement methods, which is one of the main results of this paper.

Matroid Representation for Indeterminate Trusses

The first step of representing an indeterminate truss by a matroid is to represent it by a resistance graph. Let G_R be the resistance graph representation of the indeterminate truss and $\mathbf{Q}(G)$ its scalar cutset matrix. The scalar cutset matrix defines the matroid $\mathbf{M}_Q = \langle \mathbf{S}, \mathbf{F}^I\rangle$, where $\mathbf{S}$ is the set of columns of $\mathbf{Q}(G)$ and $\mathbf{F}^I$ is a family of all linearly independent subsets of $\mathbf{S}$. The subscript Q in $\mathbf{M}_Q$ is used to emphasize that the matroid corresponds to the scalar cutset matrix $\mathbf{Q}$. Each element of $\mathbf{M}_Q$ is a scalar cutset matrix column that in its turn corresponds to a truss element, which can be rod, external reaction, or external force. An example of a truss with its corresponding matroid is given in Figs. 8(a and d), respectively.



$$
\mathbf{Q}(G)=\begin{matrix} & \begin{matrix}1 & 2 & 3 & P\end{matrix} \\ \begin{matrix}1x\\1y\end{matrix} & \begin{pmatrix}\cos(135°) & \cos(90°) & \cos(45°) & -\cos(0°)\\ \sin(135°) & \sin(90°) & \sin(45°) & -\sin(0°)\end{pmatrix}\end{matrix}
$$

$$
=\begin{pmatrix}-1/\sqrt{2} & 0 & 1/\sqrt{2} & -1\\ 1/\sqrt{2} & 1 & 1/\sqrt{2} & 0\end{pmatrix}
$$

(c)

M=<**S**,**F**^{I}>

S={1,2,3,P}

F^I = {{1}, {2},{3},{P},{1,2}, {1,3},{1,P}, {2,3},{2,P},{3,P}}.

(d)

FIG. 8. Example of Truss and Its Corresponding Matroid: (a) Truss; (b) Graph G; (c) Scalar Cutset Matrix $\mathbf{Q}(G)$; (d) Matroid

Structural Interpretation of Matroid Components

Dependent Sets of $\mathbf{M}_Q$

The flow law for G_R is given by

$$\mathbf{Q}(G)\cdot\mathbf{F}=\mathbf{0} \tag{20}$$

where $\mathbf{F}$ = vector of force scalar values acting in the truss elements. Therefore, the nonzero entries of the vector $\mathbf{F}$ define a set of linearly dependent columns of the scalar cutset matrix. By definition, such a set is also the set of dependent elements in the matroid $\mathbf{M}_Q$. Thus, a dependent set in $\mathbf{M}_Q$ corresponds to a set of truss elements in which internal forces can act simultaneously, i.e., the truss elements with nonzero internal forces during some state of self-stress. Such a set forms an indeterminate subset of truss rods (a subtruss).

Circuits of $\mathbf{M}_Q$

A circuit of the matroid is a minimal dependent set, i.e., removing even one of its elements results in an independent set. It is interesting to notice that this definition coincides with the definition of the simple circuit in graph theory which says that removing even one edge from a circuit violates the circuit property. Therefore, in the terminology of structures, a circuit in $\mathbf{M}_Q$ corresponds to a minimal indeterminate subtruss, which is a one-degree indeterminate subtruss. Such a subtruss has the properties of a circuit, since removing any of the rods from it will result in a determinate truss or even a mechanism.

Base of $\mathbf{M}_Q$

The base of a matroid is the maximal independent subset of $\mathbf{S}$, i.e., adding any element to the base results in a dependent set. Thus, the base in $\mathbf{M}_Q$ corresponds to a determinate subtruss that contains all the pinned joints of the truss. It is well known that adding a rod to a determinate truss without adding a pinned joint makes the truss indeterminate. For the sake of consistency with the graph representations, the base of the matroid representing the truss is chosen so that it does not contain any external forces (flow sources) acting on the truss.

Cobase of $\mathbf{M}_Q$

The cobase of $\mathbf{M}$ (i.e., the set of elements that are not in the base), is the set of external forces and redundant rods of the truss. The notation that is used in this paper for graphs, is applied also to matroids. For this reason, the base elements (the determinate subtruss elements) are represented by double lines, the cobase elements (redundant truss elements and external forces) by dashed lines, whereas the cobase elements that correspond to the external forces are both dashed and bold.

Fig. 9 shows the truss from Fig. 8, with highlighted base and cobase elements [Fig. 9(a)], and the two fundamental circuits (circuits containing only one redundant rod or only one external force) [Figs. 9(b and c)].

Circuit Matrix of $\mathbf{M}_Q$

According to the definition of a circuit in a matroid, because each fundamental circuit in $\mathbf{M}_Q$ corresponds to a set of linearly dependent columns in $\mathbf{Q}$ (i.e., for each fundamental circuit C_i), it can be written

$$\sum_{j\in c_i}\lambda_{ij}\mathbf{Q}_{\downarrow j}=\mathbf{0} \tag{21}$$

where $\mathbf{Q}_{\downarrow j}$ = jth column of matrix $\mathbf{Q}$. In the terminology of trusses, λ_{ij} is the force acting in the truss element j due to a force acting in element i.

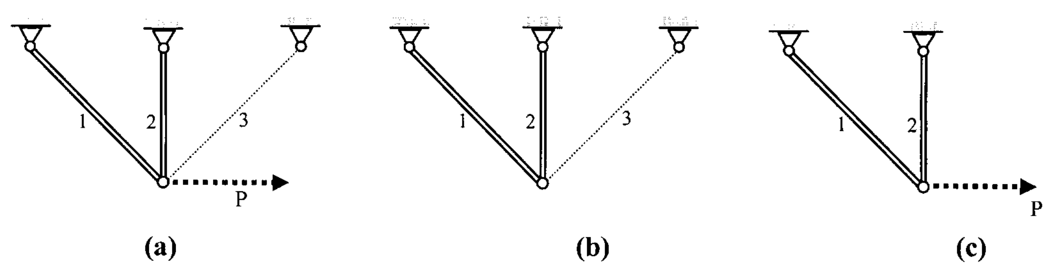


FIG. 9. Example of Fundamental Circuits in Matroid of Truss: (a) Base and Cobase of $\mathbf{M}$; (b) Fundamental Circuit Defined by Redundant Rod 3; (c) Fundamental Circuit Defined by External Force P

The set of fundamental circuits is represented by a special matrix $\mathbf{B}(\mathbf{M}_Q)$, called a circuit matrix of $\mathbf{M}_Q$. The rows of $\mathbf{B}(\mathbf{M})$ correspond to the cobase elements of $\mathbf{M}$ and the columns to all the elements of $\mathbf{M}$. An entry ij of the matroid circuit matrix is defined

$$[\mathbf{B}(\mathbf{M})]_{ij} = \lambda_{ij} \tag{22}$$

Obviously, (21) still holds, when for some i, all λ_{ij} are multiplied by the same arbitrary scalar. Therefore, it is legitimate to "normalize" the circuit matrix [i.e., to multiply the rows of $\mathbf{B}(\mathbf{M})$] so that the matrix is written as follows:

$$\mathbf{B}(\mathbf{M}) = (\mathbf{B}(\mathbf{M})_T | \mathbf{I}) \tag{23}$$

where $\mathbf{I}$ = unit matrix whose size is equal to the number of cobase elements; and $\mathbf{B}(\mathbf{M})_T$ = matrix with rows and columns corresponding to the cobase and base elements, respectively. In structural mechanics terminology, the value of $[\mathbf{B}(\mathbf{M})]_{ij}$ becomes the force in a truss rod or reaction i when a unit force is applied in a redundant element j, while in all the other redundant elements the forces are set to zero.

For example, the circuit matrix of matroid $\mathbf{M}_Q$ that represents the truss of Fig. 8 is developed as follows. For cobase elements 3 and P, equations based on (23) are written, respectively

$$\lambda_{3,1}\cdot\mathbf{Q}_{\downarrow 1} + \lambda_{3,2}\cdot\mathbf{Q}_{\downarrow 2} + \lambda_{3,3}\cdot\mathbf{Q}_{\downarrow 3} = 1\cdot\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} - 1.414\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$+ 1\cdot\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}$$

$$\lambda_{P,1}\cdot\mathbf{Q}_{\downarrow 1} + \lambda_{P,2}\cdot\mathbf{Q}_{\downarrow 2} + \lambda_{P,P}\cdot\mathbf{Q}_{\downarrow P} = -1.414\cdot\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$+ 1\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1\cdot\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}$$

Hence, the circuit matrix of $\mathbf{M}_Q$ is

$$\mathbf{B}(\mathbf{M}) = \begin{array}{c} \\ 3 \\ P \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & P \end{array} \\ \left(\begin{array}{cc|cc} 1 & -1.414 & 1 & 0 \\ -1.414 & 1 & 0 & 1 \end{array}\right) \end{array}$$

Proposition 1: Every admissible force vector $\mathbf{F}$ is a linear combination of the rows of $\mathbf{B}(\mathbf{M})$.

Proof: Forces in the determinate subtrusses (bases) are uniquely defined by the forces in the redundant elements. Moreover, each row of $\mathbf{B}(\mathbf{M})$ corresponds to the forces in the determinate subtruss yielded by a unit force acting in the corresponding redundant element. Therefore, by the superposition principle, every admissible force vector is derived by summing over all the rows of $\mathbf{B}(\mathbf{M})$ each multiplied by the force in the corresponding redundant element.

Proposition 2: The matroid potential law

$$\mathbf{B}(\mathbf{M})\cdot\mathbf{D} = \mathbf{0} \tag{24}$$

where $\mathbf{D}$ = vector of admissible scalar displacements in truss elements.

Proof: According to the definition of matroid $\mathbf{M}_Q$, each row of $\mathbf{B}(\mathbf{M})$ corresponds to a state of self-stress, which is a vector of admissible flows in G_R. On the other hand, vector $\mathbf{D}$ corresponds to a vector of admissible scalar potential differences in G_R. Thus, according to the equilibrium between the internal strain energy of the truss and the work done by the external forces (West 1993), multiplication of every row in $\mathbf{B}(\mathbf{M})$ by vector $\mathbf{D}$ is equal to zero.

Cutset Matrix of Matroid

The cutsets of a matroid are presented by a cutset matrix as explained below.

Proposition 3: The matrix $\mathbf{Q}(\mathbf{M}) = (\mathbf{I}| - \mathbf{B}^t_T)$ is the cutset matrix of matroid $\mathbf{M}_Q$, i.e., each row of $\mathbf{Q}(\mathbf{M})$ defines a fundamental cutset in the matroid.

Proof: To prove this property one has to prove that every row of $\mathbf{Q}(\mathbf{M})$ satisfies the conditions of a cutset (which are given in the Appendix).

In the second definition section of the Appendix, conditions 1 and 3 are satisfied since $\mathbf{Q}(\mathbf{M})$ contains a unit matrix whose rows are nonempty and do not contain other rows of the matrix. Condition 2 requires that for any circuit i and any cutset j the number of common elements is not equal to 1. This can be easily proved by considering forms of the circuit and the cutset matrices (Fig. 10). The number of common elements in circuit i and cutset j is the number of elements corresponding to the nonzero entires in rows i and j in the circuit and the cutset matrices, respectively. From Fig. 10 one can see that this number can be either 0 or 2 depending on whether element B_{ij} is equal to zero or not. Thus, the number of common elements in a circuit and a cutset can never be equal to 1.

Proposition 4: The orthogonality principle

$$\mathbf{Q}(\mathbf{M})\cdot\mathbf{B}^t(\mathbf{M}) = \mathbf{0} \tag{25}$$

Proof: By substituting (25) into (27) one obtains

$$\mathbf{Q}(\mathbf{M})\cdot\mathbf{B}^t(\mathbf{M}) = \begin{pmatrix} \mathbf{B}(\mathbf{M})^t_T \\ \mathbf{I} \end{pmatrix}(\mathbf{I}| - \mathbf{B}(\mathbf{M})^t_T)$$

After the multiplication one gets $\mathbf{B}(\mathbf{M})^t_T = -\mathbf{B}(\mathbf{M})^t_T = \mathbf{0}$.

Proposition 5: The matroid flow law

$$\mathbf{Q}(\mathbf{M})\cdot\mathbf{F} = \mathbf{0} \tag{26}$$

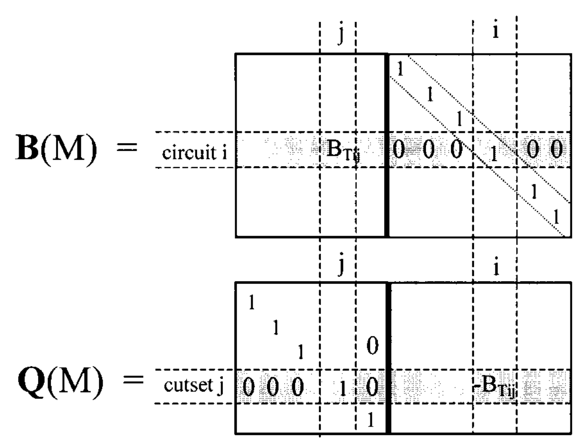
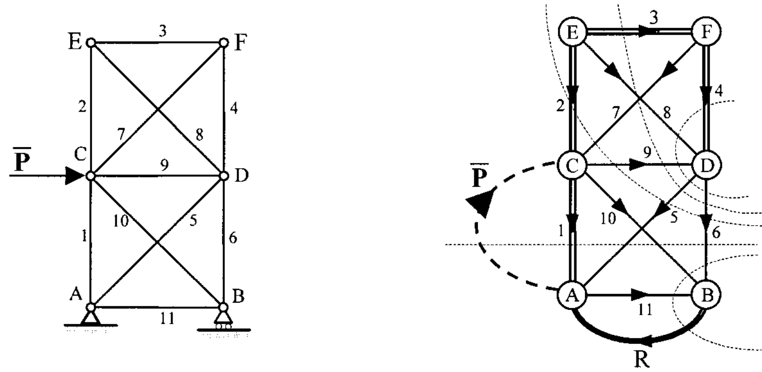


FIG. 10. Form of Circuit and Cutset Matrices



(a) (b)

$$
\overline{\mathbf{Q}} = \begin{array}{c} \\ R \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c}
\begin{array}{ccccccccccccc} R & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & P \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

(c)

$$
\mathbf{Q} = \begin{array}{c}
\begin{array}{ccccccccccccc} R & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & P \end{array} \\
\begin{pmatrix}
\cos(270^\circ) & 0 & 0 & 0 & 0 & 0 & -\cos(270^\circ) & 0 & 0 & 0 & -\cos(315^\circ) & -\cos(0^\circ) & 0 \\
\sin(270^\circ) & 0 & 0 & 0 & 0 & 0 & -\sin(270^\circ) & 0 & 0 & 0 & -\sin(315^\circ) & -\sin(0^\circ) & 0 \\
0 & \cos(270^\circ) & 0 & 0 & 0 & \cos(225^\circ) & \cos(270^\circ) & 0 & 0 & 0 & \cos(315^\circ) & 0 & \cos(0^\circ) \\
0 & \sin(270^\circ) & 0 & 0 & 0 & \sin(225^\circ) & \sin(270^\circ) & 0 & 0 & 0 & \sin(315^\circ) & 0 & \sin(0^\circ) \\
0 & 0 & \cos(270^\circ) & 0 & 0 & \cos(225^\circ) & \cos(270^\circ) & \cos(225^\circ) & 0 & -\cos(0^\circ) & 0 & 0 & 0 \\
0 & 0 & \sin(270^\circ) & 0 & 0 & \sin(225^\circ) & \sin(270^\circ) & \sin(225^\circ) & 0 & -\sin(0^\circ) & 0 & 0 & 0 \\
0 & 0 & 0 & \cos(0^\circ) & 0 & -\cos(225^\circ) & -\cos(270^\circ) & -\cos(225^\circ) & \cos(315^\circ) & \cos(0^\circ) & 0 & 0 & 0 \\
0 & 0 & 0 & \sin(0^\circ) & 0 & -\sin(225^\circ) & -\sin(270^\circ) & -\sin(225^\circ) & \sin(315^\circ) & \sin(0^\circ) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cos(270^\circ) & -\cos(225^\circ) & -\cos(270^\circ) & 0 & \cos(315^\circ) & \cos(0^\circ) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sin(270^\circ) & -\sin(225^\circ) & -\sin(270^\circ) & 0 & \sin(315^\circ) & \sin(0^\circ) & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

(d)

FIG. 11. Example of Truss and Its Corresponding Graph and Matrices: (a) Truss; (b) Graph G; (c) Vector Cutset Matrix; (d) Scalar Cutset

Proof: By Proposition 1 every admissible force vector $\mathbf{F}$ is a linear combination of $\mathbf{B}(\mathbf{M})$ rows. According to Proposition 4, $\mathbf{Q}(\mathbf{M})$ is orthogonal to $\mathbf{B}(\mathbf{M})$, hence it is orthogonal to every linear combination of its rows, i.e., $\mathbf{F}$.

Because of the validity of Propositions 1–5, the flow, potential, and orthogonality laws are all valid for matroid $\mathbf{M}$. Such a matroid is called a resistance matroid. Since (13) was derived using only these properties of the resistance graph, it is also valid for the matroid $\mathbf{M}_Q$. Substituting (12), the matrices corresponding to $\mathbf{M}_Q$ instead of those corresponding to G, yields

$$
(\mathbf{B}(\mathbf{M})_{C'R} \cdot \mathbf{R}_R \cdot \mathbf{B}(\mathbf{M})^t_{C'R}) \cdot \mathbf{F}_{C'} = -(\mathbf{B}(\mathbf{M})_{C'R} \cdot \mathbf{R}_R \cdot \mathbf{B}(\mathbf{M})^t_{PR}) \cdot \mathbf{F}_P - \mathbf{B}_{C'P} \cdot \mathbf{D}_D \quad (27)
$$

Example of Applying RCM in Resistance Matroid to Indeterminate Truss

This example will be solved using the equations derived above. It can be recognized as the well-known force method (Fig. 11). A base (statically determinate subtruss) is obtained by removing from the truss the redundant rods 7 and 10. Hence the cobase elements of the resistance matroid representing the truss are 7, 10, and P, whereas the latter is the flow source. The circuit matrix is now built by calculating three self-stresses each having a unit force in one of the cobase elements.

The matroid circuit matrix is

$$
\mathbf{B}_M = \begin{array}{c} \\ 7 \\ 10 \\ P \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 8 & 9 & 11 & 7 & 10 & P \end{array} \\
\left(\begin{array}{c|c|c|c|c|c|c|c|c|c|cc}
0 & -0.707 & -0.707 & -0.707 & 0 & 0 & 1 & -0.707 & 0 & 1 & 0 & 0 \\
-0.707 & 0 & 0 & 0 & 1 & -0.707 & 0 & -0.707 & -0.707 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -1.414 & 1 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right)
\end{array}
$$

$$
\mathbf{B}_{C'R} = \begin{array}{c} \\ 7 \\ 10 \end{array}
\begin{array}{c}
\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 8 & 9 & 11 & 7 & 10 \end{array} \\
\left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
0 & -0.707 & -0.707 & -0.707 & 0 & 0 & 1 & -0.707 & 0 & 1 & 0 \\
-0.707 & 0 & 0 & 0 & 1 & -0.707 & 0 & -0.707 & -0.707 & 0 & 1
\end{array}\right)
\end{array}
$$

$$
\mathbf{B}_{PR} = \begin{array}{c} \\ P \end{array}
\begin{array}{c}
\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 8 & 9 & 11 & 7 & 10 \end{array} \\
\left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
0 & 0 & 0 & 0 & -1.414 & 1 & 0 & 1 & 0 & 0 & 0
\end{array}\right)
\end{array}
$$

Now, (27) turns into

$$\begin{pmatrix} 4.823 & 0.5 \\ 0.5 & 4.823 \end{pmatrix} \begin{pmatrix} F_7 \\ F_{10} \end{pmatrix} = \begin{pmatrix} 0.707 \\ 3.411 \end{pmatrix} \cdot P$$

Substituting $P = 1$ gives the following solution to the equations $F_7 = 0.074$ and $F_{10} = 0.699$:

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_8 \\ F_9 \\ F_{11} \\ F_7 \\ F_{10} \\ P \end{bmatrix} = \bar{\mathbf{F}} = \mathbf{B}(\mathbf{M})^t \bar{\mathbf{F}}_c = \begin{pmatrix} 0 & -0.707 & -0.707 & -0.707 & 0 & 0 & 1 & -0.707 & 0 & 1 & 0 & 0 \\ -0.707 & 0 & 0 & 0 & 1 & -0.707 & 0 & -0.707 & -0.707 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1.414 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\cdot \begin{pmatrix} 0.074 \\ 0.699 \\ 1 \end{pmatrix} = \begin{bmatrix} -0.494 \\ -0.0523 \\ -0.0523 \\ -0.0523 \\ -0.714 \\ 0.506 \\ 0.074 \\ 0.453 \\ -0.494 \\ 0.074 \\ 0.699 \\ 1 \end{bmatrix}$$

Now, using the relations given in (8), the forces in the remainder of the truss elements can be obtained.

GENERAL PERSPECTIVE ON REPORTED APPROACH AND ITS APPLICATIONS

As was shown in this paper, applying combinatorial representations to structural analysis enables one to get a general perspective, providing some new and global results. Furthermore, taking advantage of the fact that the same approach has been applied to different fields, the dualism between determinate trusses and mechanisms has been established, due to the dualism connection between their corresponding representations. This dualism opened a new avenue in research and made it possible to obtain new applications, some of which are mentioned below. The dualism gave new insight into the applicability of the approach underlying the current paper, whereas the main issue of the paper itself was to establish the theoretical foundation for future applications in structural analysis. From the contribution of establishing the truss-mechanism mutual dualism, one can conclude the importance of deriving the dualism between force and displacement methods, which is reported in this paper.

Dualism between Determinate Trusses and Mechanisms

Fig. 12 gives the trace of how the truss-mechanism dualism was derived, and an example of a mechanism and its dual truss appears in Fig. 13. In this case, the dualism is based on the fact that the cutset matrix of graph G is actually the circuit matrix of its dual graph G^*. When the flow vector $\bar{\mathbf{F}}$ of G is equal to the potential difference vector $\bar{\boldsymbol{\Delta}}$ of G^*, the flow law becomes the potential law in the dual graph and vice versa. Since G corresponds to a determinate truss that satisfies the flow law, and G^* to a mechanism that satisfies the potential law, the mutual dualism is derived.

This new mutual dualism opens up new possibilities for applications, some of which have already been carried out and are listed below, whereas many others are currently being ex-

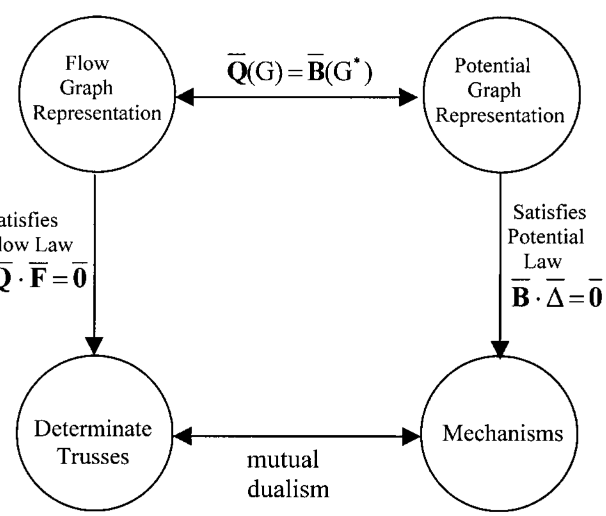


FIG. 12. Schematic Explanation of Mutual Dualism between Determinate Trusses and Mechanisms

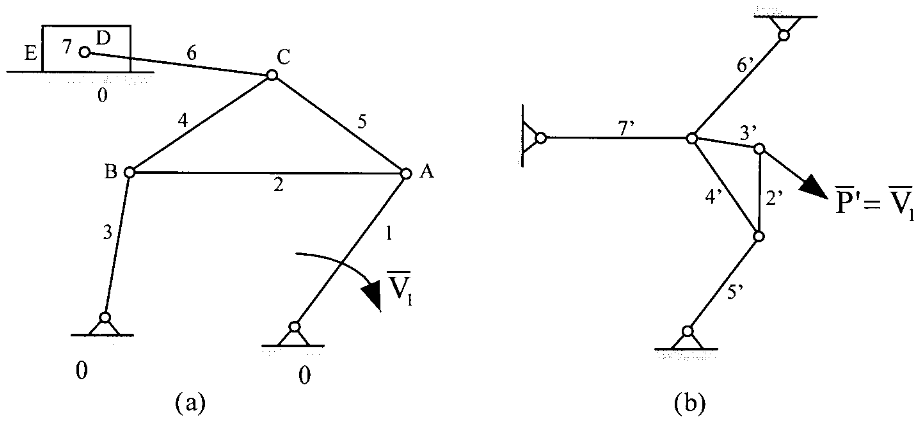


FIG. 13. Example of Mechanism and Its Dual Truss: (a) Mechanism; (b) Dual Truss

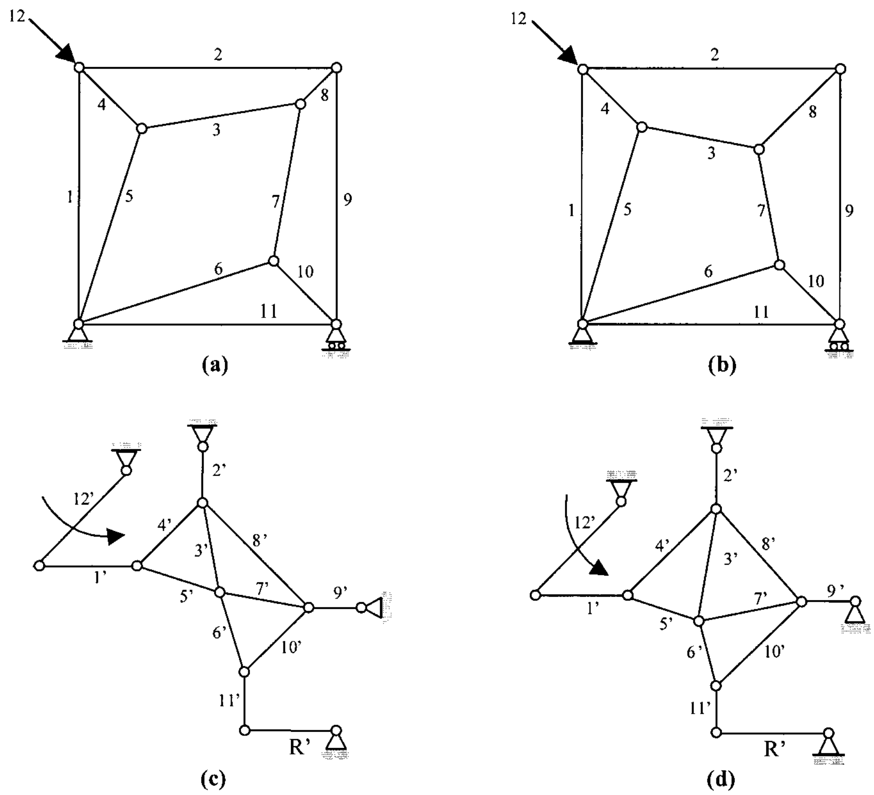


FIG. 14. Example of Stable and Nonstable Trusses and Their Dual Mechanisms: (a) Stable Truss; (b) Nonstable Truss; (c and d) Corresponding Dual Mechanisms

amined. This new connection enables one to use knowledge, theorems, and algorithms from one engineering field in the other, and vice versa. For example, methods like Henneberg, Maxwell-Cremona, and others from structural mechanics were applied to machine theory. On the other hand, image-velocity, decomposition to kinematical groups (Assur groups) are now being applied to structural mechanics.

In addition, this connection was found to be useful in artificial intelligence representations, and reaffirmed the claim of Simon (1981) who wrote: ''Solving a problem simply means representing it so as it make the solution transparent.'' In Figs. 14(a and b) two determinate trusses are given, whereas only the first one is stable. Reaching this conclusion without performing calculations is not easy even for experts in mechanical engineering. On the other hand, considering the mechanisms dual to these trusses makes the task easier. Fig. 14(c) shows the mechanism dual to the truss in Fig. 14(a), whereas Fig. 14(d) shows the mechanism dual to the truss in Fig. 14(b). In the mechanism of Fig. 14(d), links 1 and 9 are collinear, in contrast to the mechanism of Fig. 14(c). Therefore, it is easy to derive that the dual mechanism of the first truss in Fig. 14(d) is locked while the dual mechanism of the second truss is not. This makes it possible to conclude that the truss in Fig. 14(a) is not stable, whereas the truss in Fig. 14(b) is stable. This example strengthens the claim that there are properties that are hard to detect in the primal representation, whereas they are transparent in the dual.

Dualism between Force and Displacement Methods

A better insight into the approach is achieved when one notices that the approach applied to derive the dualism between trusses and mechanisms was also applied to derive the dualism between the force and displacement methods, as shown in Fig. 15. This new relation was derived on the basis of the duality connection between CCM and RCM in the matroid resistance representation. On the basis of the results achieved by establishing the mutual dualism between trusses

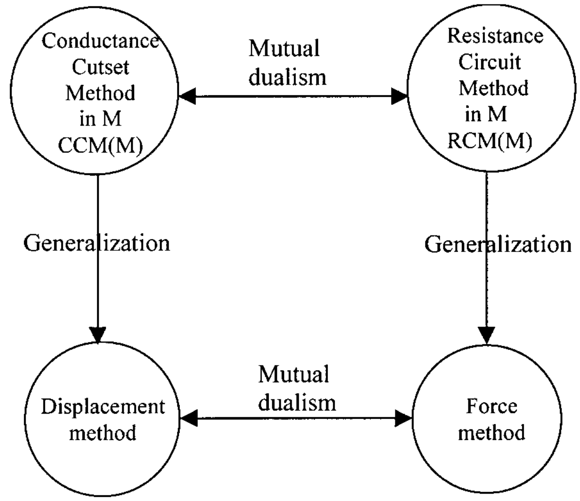


FIG. 15. Schematic Explanation of Mutual Dualism between Displacement and Force Methods

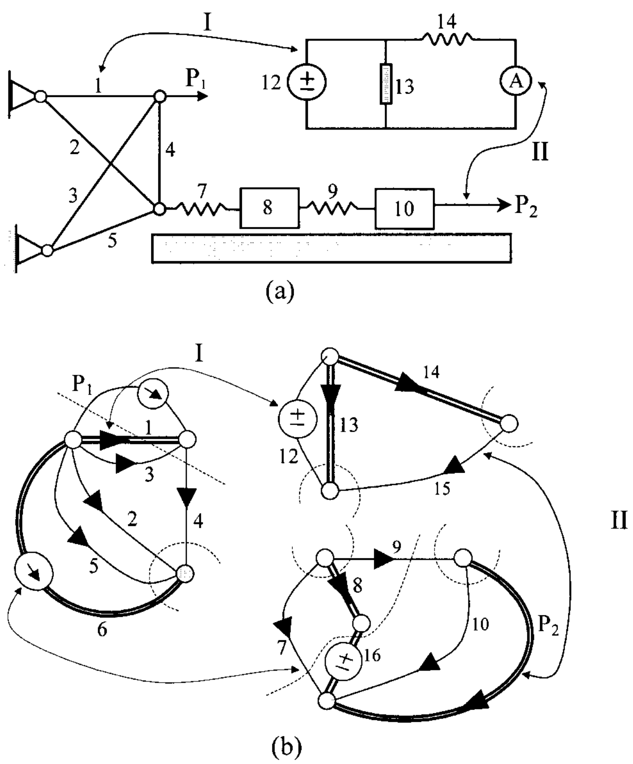


FIG. 16. (a) Integrated Multidisciplinary System; (b) Corresponding Graph

and mechanisms, one can foresee the potential offered by the dualism between the force and displacement methods.

Practicality of This Approach

In the current paper, the resistance graph has been applied to structural analysis. Earlier, this representation has also been applied to mass-spring-damper dynamic systems (Shai and Preiss 1999b), electrical systems (Shai 2001b), and other systems. Hence, an important practical application arises enabling one to deal with integrated multidisciplinary systems consisting of elements from the above variety of fields interacting with one another. An example for such a system is presented in Fig. 16. More details on this application possibility can be found in Reich and Shai (2000).

Applying Combinatorial Representations to Education

As explained above, the combinatorial representations enable one to achieve a new insight into different disciplines. This idea was applied to the development of a new teaching method by which students are first taught the combinatorial representations, and only then do they learn the engineering material. Due to this new way of teaching, students have learned the engineering material from a new multidisciplinary perspective. For example, analysis of mechanisms and determinate trusses were taught on the basis of potential and flow graph representations. Since the last representations are dual, the material was taught simultaneously, thus when a student faced a problem in a mechanism, he was able to transform it to a determinate truss. By doing that he used knowledge both from machine theory and structural mechanics, which helped him in the studying process. Until now, more than 250 students have already participated in this course and related methodological material is being prepared for high school teachers. More details can be found in Shai and Preiss (1994).

This approach can also be used for artificial intelligence-based reasoning, where this time the reasoning is performed on the basis of CR. Since the mathematical foundation of the representations is discrete mathematics, it is easy to implement the representations on a computer. Moreover, the reasoning made upon the representations can use additional knowledge, called "embedded engineering knowledge." This approach opens up a wide variety of additional applications, but those are beyond the scope of the current paper. Examples and more details can be found in Shai and Preiss (1999b) and Shai (2001b).

CONCLUSIONS

This paper started with a brief introduction to the general approach, the idea of which is to build combinatorial mathematical models called combinatorial representations (CR), with which to represent various engineering systems. The current paper introduced only those representations that were applied to structural analysis, namely flow graph, resistance graph, and resistance matroid representations. The flow graph was applied to the analysis of determinate trusses, whereas the resistance graph and resistance matroid were applied to indeterminate trusses. The last two representations contain two mutually dual methods called CCM and RCM. When the resistance graph was applied to represent indeterminate trusses, CCM was proven to be equivalent to the known displacement method. When the matroid representation was applied to represent indeterminate trusses, RCM was shown to be equivalent to the known force method. Based on the mutual connection between CCM and RCM, it was derived that the displacement and force methods are dual methods.

The contribution of this approach to structural analysis is not only theoretical but also practical. It was explained in the present paper that from the fact that the same CR were used for structural analysis and also for various engineering systems, a new way of representing multidisciplinary systems has arisen. In addition, the conditions between the CR opened a new avenue for the use of algorithms, methods, and theorems from machine theory in structural analysis and vice versa. This approach also enables a new teaching method, by which students first learn the CR and only then the engineering material. By doing that, students gain a general perspective on different engineering fields and comprehend the new relations between the fields that today are considered to be different.

APPENDIX. BASICS OF MATROID THEORY

Definition: If one denotes **S** to be a finite set, and **F** to be a collection of certain subsets of *S*, then the pair $\mathbf{M} = \langle \mathbf{S}, \mathbf{F}^I \rangle$ is called a matroid if the following properties are satisfied:

1. $\emptyset \in \mathbf{F}^I$.
2. If $\mathbf{X} \in \mathbf{F}^I$ and $\mathbf{Y} \subseteq \mathbf{X}$ then $\mathbf{Y} \in \mathbf{F}^I$ must also hold.
3. If $\mathbf{X} \in \mathbf{F}^I$ and $\mathbf{Y} \in \mathbf{F}^I$ and $|\mathbf{X}| > |\mathbf{Y}|$ then there exists an element $x \in \mathbf{X} - \mathbf{Y}$, so that $\mathbf{Y} \cup \{x\} \in \mathbf{F}^I$.

S is said to be the underlying set of matroid **M**. The subsets of **S** that belong to $\mathbf{F}^I$ are said to be independent subsets, otherwise they are called dependent subsets.

Example: Let **Q** be an $m \times n$ matrix. The matroid $\mathbf{M}_Q = \{\mathbf{S}_Q, \mathbf{F}^I_Q\}$ can be defined as follows:

1. The underlying set $\mathbf{S}_Q$ is the set of n column vectors of **Q**.
2. Every subset of linearly independent columns of **Q** belongs to $\mathbf{F}^I$.

Maximal independent sets of **M** (i.e., independent sets that are not contained in any other independent set of **M**) are called bases of **M**. For every base of **M** there is a corresponding cobase which is the complement of the base to **S**. It can be proved (Recski 1989) that the sizes of all the bases of a matroid are equal. Every matroid can be described also by the collection of all its bases **T**, instead of the collection of all its independent sets **F** as was done in the example above. Minimal dependent sets of **M** (i.e., dependent sets that do not contain other dependent sets) are called circuits of **M**. The collection of all the circuits of **M** is denoted by **C**.

Definition of a matroid cutset: The subset $\mathbf{X} \subseteq \mathbf{S}$ is called a cutset of **M** if and only if it satisfies the following conditions:

1. $\mathbf{X} \neq \emptyset$
2. $|\mathbf{X} \cap \mathbf{Y}| \neq 1$ for every $\mathbf{Y} \in \mathbf{C}$
3. **X** is minimal with respect to these properties

Since a base is a maximal possible set of independent elements, adding an additional element to the base turns it into a dependent set, i.e., a set that contains a circuit. Therefore, every cobase element defines exactly one circuit that contains the element itself and all the other elements are from the base only. Such a circuit is called a fundamental circuit. In graph theory terminology a base is a spanning tree over the matroid.

It can also be shown that every base element defines a unique cutset that contains the element itself while all the other elements are the cobase elements. More details about matroid theory can be found in Recski (1989).

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NOTATION

The following symbols are used in this paper:

$\underline{\mathbf{A}}$ = incidence matrix;
$\overline{\mathbf{B}}$ = vector circuit matrix;
$\mathbf{B}(\mathbf{M})$ = circuit matrix of matroid;
$\mathbf{C}$ = set of matroid circuits;
$\underline{\mathbf{D}}$ = vector of scalar displacements in truss elements;
$\dim(\overline{F})$ = dimension of forces acting in truss;
E = set of graph edges;
$e(G)$ = number of edges in graph G;
$\underline{\mathbf{F}}$ = scalar flow vector;
$\overline{\mathbf{F}}$ = flow vector;
$\mathbf{F}^I$ = independent subsets of matroid;
$\underline{F}(e)$ = value of flow in edge e;
$\overline{F}(e)$ = flow in edge e;
G = graph;
G^* = dual graph of graph G;
G_F = flow graph;
G_R = resistance graph;
G_Δ = potential graph;
$K(e)$ = scalar conductance of edge e;
$\mathbf{K}(e)$ = matrix conductance of edge e;
$\mathbf{K}_R$ = square matrix containing conductances of resistance edges of graph;
$\mathbf{M}$ = matroid;
$\mathbf{M}_Q$ = matroid defined by matrix **Q**;
$\overline{\mathbf{P}}$ = vector of flows in flow sources;
$\underline{\mathbf{Q}}$ = scalar cutset matrix;
$\mathbf{Q}$ = vector cutset matrix;

$\mathbf{Q}(\mathbf{M})$ = cutset matrix of matroid;
$\hat{\mathbf{r}}(e)$ = unit vector in direction of edge e;
$R(e)$ = scalar resistance of edge e;
$\mathbf{R}(e)$ = matrix resistance of edge e;
$\mathbf{R}_R$ = square matrix containing resistances of resistance edges of graph;
$\mathbf{S}$ = underlying set of matroid;
T = spanning tree;
T' = spanning tree without sources;
$\mathbf{T}$ = set of matroid bases;
$\underline{\mathbf{V}}$ = set of graph vertices;
$\bar{V}_i$ = relative linear velocity of link i;
$\mathbf{v}(G)$ = number of vertices in graph G;
$\mathbf{0}$ = zero matrix;
$\underline{\boldsymbol{\Delta}}$ = scalar potential difference vector;
$\bar{\boldsymbol{\Delta}}$ = potential difference vector;
$\bar{\Delta}(e)$ = potential difference in edge e; and
$\pi(i)$ = potential of vertex i.